III.—*On the Mean Results of Observations. By the* Rev. HUMPHREY LLOYD, *D. D.*, PRESIDENT; *F. R. S.; Hon. F. R. S. E.; Corresponding Member of the Royal Society of Sciences at Gottingen; Honorary Member of the American Philosophical Society, of the Batavian Society of Sciences, and of the Society of Sciences of the Canton de Vaud, &c.*

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1. THE problem in which it is sought to determine the daily mean values of the atmospheric temperature or pressure, from a limited number of observed values, is one of fundamental importance in meteorology; and, accordingly, many solutions of it have been proposed by meteorologists. These solutions are derived, for the most part, from the known laws of the diurnal variation of these elements. Many of them are accordingly applicable only to the particular cases considered; while for others, which are really general in their nature, that generality is not established. It is the object of the following investigation to supply this deficiency, and to show in what manner the daily and yearly means may be obtained in all the periodical functions with which we are concerned in magnetism and meteorology.

2. It is known that the mean value of any magnetical or meteorological element, for any day, may be obtained, approximately, by taking the *arithmetical mean* of any number of *equidistant* observed values; the degree of approximation, of course, increasing with the number. A somewhat more exact mean may be deduced, as has been shown by COTES and KRAMP, by combining the equidistant observed values in a different manner; and GAUSS has given a method, whereby the values of the integral, $\int_{-a}^{+a} U dx$, may be obtained with still greater accuracy from the observed values of the ordinate, U, corresponding to

certain definite abscissæ.* But in the case of periodical functions, it will appear from what follows that the refinement of COTES is unnecessary; and, in the case under consideration, there are practical reasons of another kind for adhering to the method of equidistant observations, and which, therefore, deprive us of the advantages of GAUSS's method.

3. Any periodical function U, of the variable x, may be represented by the series

$$U = A_0 + A_1 \sin (x + a_1) + A_2 \sin (2x + a_2) + A_3 \sin (3x + a_3) + \&c.,$$

in which the first term, A_0, is the mean value of the ordinate U, and is expressed by the equation

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} U dx.$$

This is the quantity whose value is sought in the present investigation.

It is obvious that the values of U return again in the same order and magnitude when x becomes $x + 2\pi$; so that if $x = at$, the period is represented by $\frac{2\pi}{a}$. If then 2π be divided into n equal parts, so that the abscissæ of the points of division are x, $x + \frac{2\pi}{n}$, $x + \frac{4\pi}{n}$, &c., $x + \frac{2(n-1)\pi}{n}$, the sum of the corresponding ordinates will be

$$\Sigma(U) = nA_0 + A_1 \Sigma \sin\left(x + \frac{2i\pi}{n} + a_1\right) + A_2 \Sigma \sin\left\{2\left(x + \frac{2i\pi}{n}\right) + a_2\right\}$$
$$+ A_3 \Sigma \sin\left\{3\left(x + \frac{2i\pi}{n}\right) + a_3\right\} + \&c.$$

in which i denotes any one of the series of integer numbers, from 0 to $n - 1$ inclusive. The multiplier of A_m, in the general term of this series, is

$$\Sigma \sin\left\{m\left(x + \frac{2i\pi}{n}\right) + a_m\right\}$$
$$= \sin(mx + a_m)\, \Sigma \cos \frac{2im\pi}{n} + \cos(mx + a_m)\, \Sigma \sin \frac{2im\pi}{n}.$$

* *Commentationes Societatis Regiæ Scientiarum Gottingensis*, tom. iii.

But, when m is not a multiple of n,

$$\Sigma \cos \frac{2im\pi}{n} = 0, \quad \Sigma \sin \frac{2im\pi}{n} = 0;$$

and, therefore, the preceding term vanishes. When m is a multiple of n,

$$\Sigma \cos \frac{2im\pi}{n} = n, \quad \Sigma \sin \frac{2im\pi}{n} = 0;^*$$

and accordingly the term is reduced to

$$n \sin (mx + \alpha_m).$$

Hence, all the terms of the series vanish, excepting those in which $m = kn$, k being any number of the natural series, and there is

$$\frac{1}{n}\Sigma(U) = A_0 + A_n \sin (nx + \alpha_n) + A_{2n} \sin (2nx + \alpha_{2n}) + \&c.$$

That is, the arithmetical mean of the n equidistant ordinates is equal to the sum of the terms of the original series of the order kn, whatever be the value of x.

The original series for U being always convergent, the derived series, which expresses the value of $\frac{1}{n}\Sigma(U)$, will be much more so; and, when the

* These results are easily established. The roots of the equation $y^n - 1 = 0$, being comprised in the formula $\cos \frac{2i\pi}{n} + \sqrt{(-1)} \sin \frac{2i\pi}{n}$, the m^{th} power of any one of these roots is $\cos \frac{2im\pi}{n} + \sqrt{(-1)} \sin \frac{2im\pi}{n}$; and the sum of the m^{th} powers of the roots is

$$\Sigma \cos \frac{2im\pi}{n} + \sqrt{(-1)}\ \Sigma \sin \frac{2im\pi}{n}.$$

Now, when m is not a multiple of n, this sum $= 0$, and therefore

$$\Sigma \cos \frac{2im\pi}{n} = 0, \quad \Sigma \sin \frac{2im\pi}{n} = 0;$$

as above. When m is a multiple of n, the sum of the m^{th} powers of the roots $= n$, and therefore

$$\Sigma \cos \frac{2im\pi}{n} = n, \quad \Sigma \sin \frac{2im\pi}{n} = 0.$$

This demonstration seems preferable to that derived from the general formulæ for the sum of the sines and cosines of arcs in arithmetical progression, which, in the latter of the two cases above mentioned, lead to illusory results.

number n is sufficiently great, we may neglect all the terms after the first. Hence, approximately, $A_0 = \frac{1}{n} \Sigma (U)$.

The error of this result will be expressed by the second term of the series, $A_n \sin (nx + \alpha_n)$, the succeeding terms being, for the same reason, disregarded in comparison; and accordingly the limit of error will be A_n. Thus, when the period in question is a day, we learn that the *daily mean value* of the observed element will be given by the mean of *two* equidistant observed values, nearly, when A_2 and the higher coefficients are negligible; by the mean of *three*, when A_3 and the higher coefficients are negligible; and so on.

4. The coefficient A_2 is small in the series which expresses the diurnal variation of temperature; and, consequently, the curve which represents the course of this variation is, nearly, the curve of sines. In this case, then, the mean of the temperatures at any two equidistant or *homonymous* hours is, nearly, the mean temperature of the day. The same thing holds with respect to the annual variation of temperature; and the mean of the temperatures of any two equidistant months is, nearly, the mean temperature of the year. These facts have been long known to meteorologists.

5. The coefficient A_3 is small in *all* the periodical functions with which we are concerned in magnetism and meteorology; and, therefore, the daily and yearly mean values of these functions will be given, approximately, by the mean of any *three* equidistant observed values.

In order to establish this, as regards the daily means, I have calculated the coefficients of the equations which express the laws of the mean diurnal variation of the temperature, the atmospheric pressure, and the magnetic declination, as deduced from the observations made at the Magnetical Observatory of Dublin during the year 1843. The observations were taken every alternate hour during both day and night; and the numbers employed in the calculation are the yearly mean results corresponding to the several hours. The origin of the abscissæ is taken at midnight.

6. The following is the equation of the diurnal variation of temperature:

$$\begin{aligned} U - A_0 = &+ 3^\circ{\cdot}60 \sin (x + 239^\circ{\cdot}0) + 0^\circ{\cdot}70 \sin (2x + 67^\circ{\cdot}2) \\ &+ 0^\circ{\cdot}26 \sin (3x + 73^\circ{\cdot}5) + 0^\circ{\cdot}03 \sin (4x + 102^\circ{\cdot}7) \\ &+ 0^\circ{\cdot}14 \sin (5x + 258^\circ{\cdot}6) + 0^\circ{\cdot}09 \sin (6x + 180^\circ). \end{aligned}$$

Hence the error committed, in taking the mean of the temperatures at any two equidistant hours as the mean temperature of the day, is expressed nearly by the term

$$0^\circ{\cdot}70 \sin (2x + 67^\circ{\cdot}2) ;$$

and consequently cannot exceed $0^\circ{\cdot}70$. To obtain the pairs of homonymous hours, whose mean temperature corresponds most nearly with that of the day, we have only to make $\sin (2x + 67^\circ{\cdot}2) = 0$; which gives for x the values

$$x = 56^\circ{\cdot}4, \quad x = 146^\circ{\cdot}4,$$

corresponding to the times

$$t = 3^h\, 46^m, \quad t = 9^h\, 46^m.$$

Accordingly, the best pairs of homonymous hours, so far as this problem is concerned, are $3^h\, 46^m$ A. M. and $3^h\, 46^m$ P. M., or $9^h\, 46^m$ A. M. and $9^h\, 46^m$ P. M.

The error committed, in taking the mean of the temperatures at any *three* equidistant hours as the mean temperature of the day, is, very nearly,

$$+\, 0^\circ{\cdot}26 \sin (3x + 73^\circ{\cdot}5) ;$$

and cannot therefore exceed $0^\circ{\cdot}26$. The best hours are those in which the angle, in the preceding expression, is equal to 180° or 360°. The corresponding values of x are

$$x = 35^\circ{\cdot}5, \quad x = 95^\circ{\cdot}5 ;$$

whence

$$t = 2^h\, 22^m, \quad t = 6^h\, 22^m.$$

Accordingly, the best hours of observation are

$2^h\, 22^m$ A. M., $10^h\, 22^m$ A. M., $6^h\, 22^m$ P. M. ;

and

$6^h\, 22^m$ A. M., $2^h\, 22^m$ P. M., $10^h\, 22^m$ P. M.

By taking the mean of any *four* equidistant observed values, the limit of error will, of course, be less. Its amount, which is the coefficient of the fourth term of the preceding formula, is only $0^\circ{\cdot}03$; and, accordingly, the mean temperature of the day is inferred from the temperatures observed at *any* four equidistant hours with as much precision as can be desired.

7. The law of the diurnal variation of the atmospheric pressure is contained in the following equation :

$$U - A_0 = + \cdot 0024 \sin (x + 244^\circ \cdot 3) + \cdot 0089 \sin (2x + 144^\circ \cdot 4) \\ + \cdot 0008 \sin (3x + 27^\circ \cdot 9) + \cdot 0006 \sin (4x + 78^\circ \cdot 5) \\ + \cdot 0001 \sin (5x + 228^\circ \cdot 7) + \cdot 0002 \sin (6x + 180^\circ).$$

The second term in this formula being the principal one, the mean of the pressures observed at any *two* equidistant hours, so far from approaching the mean daily pressure, may recede from it by the greatest possible amount within the limits of the diurnal variation. The error committed, in taking the mean of the pressures observed at *three* equidistant hours as the mean daily pressure, is, very nearly,

$$+ \cdot 0008 \sin (3x + 27^\circ \cdot 9) ;$$

and cannot therefore exceed ·0008. It is needless to inquire into the least value of this quantity, which is in all cases less than the probable error.

8. The law of the diurnal variation of the magnetic declination is expressed by the equation

$$U - A_0 = + 3' \cdot 29 \sin (x + 65^\circ \cdot 7) + 2' \cdot 08 \sin (2x + 224^\circ \cdot 5) \\ + 0' \cdot 63 \sin (3x + 71^\circ \cdot 7) + 0' \cdot 30 \sin (4x + 237^\circ \cdot 5) \\ + 0' \cdot 13 \sin (5x + 114^\circ \cdot 7) ;$$

the coefficient of the last term being evanescent. Hence the error to which we are liable, in taking the mean of the declinations observed at any three equidistant hours as the mean of the day, is, very nearly,

$$+ 0' \cdot 63 \sin (3x + 71^\circ \cdot 7) ;$$

and cannot exceed $0' \cdot 63$. This term vanishes, and the mean of the three observed values will deviate from the true daily mean, by an amount less than the errors of observation, when

$$x = 36^\circ \cdot 1, \text{ or, } x = 96^\circ \cdot 1 ;$$

that is, when

$$t = 2^h\, 25^m, \text{ or, } t = 6^h\, 25^m.$$

Accordingly, the best hours of observation, for the elimination of the diurnal variation of the declination, are

$$2^h\, 25^m \text{ A. M.}, \quad 10^h\, 25^m \text{ A. M.}, \quad 6^h\, 25^m \text{ P. M.} ;$$

and

$$6^h\, 25^m \text{ A. M.}, \quad 2^h\, 25^m \text{ P.M.}, \quad 10^h\, 25^m \text{ P. M.} ;$$

which coincide, almost exactly, with the best hours for the determination of the mean temperature.

By taking the mean of the declinations observed at any *four* equidistant hours, as the mean of the day, the limit of error is reduced to 0′·30.

9. It appears from the preceding, that *any* three equidistant observations are sufficient to give the daily mean values (and, therefore, also the monthly and yearly mean values) for each of these elements, with nearly the requisite precision ; and that, by a suitable choice of the hours, the degree of accuracy may be augmented as much as we please. But, in determining the *particular* hours for a continuous system of observations, this should not be made the primary ground of selection. The error of the daily means being in all cases reduced within narrow limits by the method already explained, we should choose the particular hours which correspond nearly to the maxima and minima of the observed elements, so as to obtain also the *daily ranges.* This condition will be fulfilled in the case of the *magnetic declination,* very nearly, by the hours

6 A. M., 2 P. M., 10 P. M. ;

which will, moreover, give nearly the maximum and minimum of *temperature,* and of the *tension of vapour*, together with the maximum pressure of the *gaseous atmosphere.** And, if we add the intermediate hours, 10 A. M. and 6 P. M., we shall have, nearly, the principal maxima and minima of the two other magnetic elements. Accordingly, for a limited system of magnetical and meteorological observations, at places for which the epochs of maxima and minima do not differ much from those at Dublin, the best hours of observation appear to be

6 A. M., 10, 2 P. M., 6, 10.

The conditions of the problem are altered, if at any place the laws of the diurnal variation have been already obtained from a more extended system of

* The ternary combination above proposed possesses the further advantage of coinciding, nearly, with one of those deduced above, as the *most* favourable for the determination of the *mean* temperature and *mean* declination. The errors of the resulting means are found by making $x = 90^\circ$ in the third terms of the general formulæ ; and we thus find the error of temperature $= -0^\circ{\cdot}07$, while that of the declination $= -0'{\cdot}20$.

observations. In this case the mean of the day may be inferred from observations taken at *any hours* whatever, by the addition of a known correction; and the hours of observation should therefore be chosen chiefly, if not exclusively, with reference to the diurnal range of the observed elements.

10. The next question which presents itself for consideration, with respect to the daily means, is one which affects more nearly the reduction of the observations hitherto made at Dublin. In the extended system prescribed by the Council of the Royal Society in 1839, and followed at the Magnetical Observatory of Dublin during the four years commencing with 1840, observations were directed to be taken twelve times, at equal intervals, throughout the day,—namely, at the even hours of Gottingen mean time. In a system of observations so frequent, and extending over so considerable a time, blanks must unavoidably occur; and the question which presents itself here is,—in what way are the daily means to be deduced in such a case?

It has been shown that the effect of the *regular* diurnal variation may be nearly eliminated, and the mean of the day obtained, by taking the mean of three equidistant observed values. For the elimination of the *irregular* changes, however, the number of observations combined should be as great as possible; and in the case of the magnetic elements, in which these changes are often very considerable, this condition is an important one.

Now it is obvious that the twelve results of any day may be resolved into *two* groups of *six* equidistant results, or into *three* groups of *four*, or into *four* groups of *three*. Hence, when *one* result is wanting in the day, the mean may be inferred either from one group of six results, from two groups of four results, or from three of three. The last of these combinations, containing nine separate results, is, of course, to be preferred. When *two* results are wanting, the mean may be inferred from one group of four results, or from two groups of three; of which the latter combination, containing six results, is to be preferred. When *three* results are wanting, the mean of the day can only be inferred (in general) from one group of three; and when more than three are wanting, that mean cannot be generally obtained.

11. What has been said above applies to the irregular changes of short period, such, especially, as those to which the magnetic elements are subject. But there are also irregular changes of longer duration (as, for ex-

ample, those produced in the atmospheric pressure by the passage of the greater aerial waves), which complicate the problem, inasmuch as a different process is required for their elimination.

In the reduction of the magnetical and meteorological observations made at the Observatory of Dublin, the *civil* day is adopted; and the observations being made at the *odd* hours of Dublin mean time, very nearly, the *epoch* of the mean of all the twelve results is *mean noon.* But in the case of deficient observations, the epoch of the mean, inferred from the remaining observations, may deviate one or more hours from noon; and its amount, therefore (as compared with the mean *reduced to noon*), is affected by an error equal to the change which the observed element undergoes in that time. In the case of the atmospheric pressure, this error is often very considerable, and much exceeds that due to the changes of whose elimination we have hitherto spoken.

The law of the changes here referred to being unknown, we can only deal with them on the assumption that their course is uniform throughout the space of a day; and this assumption will, probably, seldom err much from the truth. Upon this principle, the effect of the irregular change will be eliminated by taking the mean of two or more results *equidistant from noon* (that is, the mean of a forenoon and afternoon result corresponding to the hours x and $12 - x$, or any combination of such means); and we have only to consider in what manner this process can be combined with the elimination of the *regular* diurnal change.

Let the mean of the *four* equidistant observed values commencing with the n^{th} hour be denoted, for brevity, by IV_n; then the *epochs* of the means IV_1, IV_3, IV_5, are 10 A.M., noon, and 2 P.M., respectively; so that the two conditions are satisfied by the combinations

$$\tfrac{1}{2}(\mathrm{IV}_1 + \mathrm{IV}_5), \text{ and } \mathrm{IV}_3.$$

In like manner, the means of any *three* equidistant observed values being denoted by III_n, the epochs of the means III_1, III_3, III_5, III_7, are 9 A.M., 11, 1 P.M., and 3 respectively; so that both conditions are satisfied by the combinations

$$\tfrac{1}{2}(\mathrm{III}_1 + \mathrm{III}_7), \text{ and } \tfrac{1}{2}(\mathrm{III}_3 + \mathrm{III}_5).$$

12. When, from the number and disposition of the blanks, none of these combinations can be had, and therefore both changes (regular and irregular)

cannot be eliminated, we must attend chiefly to that which is greater in amount. For the purpose of comparing their magnitude, I have taken the differences of the successive daily means, for the declination, the atmospheric pressure, and the temperature, as deduced from the observations of the year 1843 ; and have calculated the square root of the mean of the squares of these differences. The results, which may be taken as the measures of the irregular changes from day to day, are the following:

Mean Fluctuation from Day to Day.

Magnetic declination, . . . Fluctuation = $1'\cdot04$.
Atmospheric pressure, ... 0·214.
Atmospheric temperature, ... $3^{\circ}\cdot07$.

Similarly, if we take the differences of the yearly means corresponding to the successive hours of observation, and combine them in the same way, we obtain the mean two-hourly fluctuations, arising from the regular diurnal change. These numbers are the following:

Mean Fluctuation in two Hours.

Magnetic declination, . . . Fluctuation = $2'\cdot04$.
Atmospheric pressure, ... 0·0065.
Atmospheric temperature, ... $1^{\circ}\cdot46$.

These numbers, compared with the twelfth part of the former, serve to measure the relative magnitude of the regular and irregular changes to which the elements are subject in the same time. We thus find that, in the case of the magnetic declination, the irregular change (which is less than $\frac{1}{20}$th part of the regular) may be safely neglected; and we have only to attend to the diurnal changes, and to the irregular changes of *short* period. The daily means are, therefore, to be deduced from one of the combinations of Art. 10, giving the preference to that which contains the greatest number of individual results.

In the case of the atmospheric temperature, the irregular change (which is less than one-fifth part of the regular) is small; aud we must attend chiefly to the latter. The mean of the day is, therefore, to be inferred from one of

the combinations of Art. 10, giving the preference to those of Art. 11, whose epoch is noon.

In the case of the atmospheric pressure, on the contrary, the irregular change (which is triple the regular) is the more important. The mean of the day is, therefore, to be deduced from any combination whose epoch is noon, giving, however, the preference to one of those of Art. 11, in which the diurnal change is also eliminated.

13. I now proceed to consider the reduction of the *monthly* means, in the case of deficient observations.

For the purpose of determining the regular diurnal variation of any magnetic or meteorological element, it is necessary to take the mean of an adequate number of separate results corresponding to each hour of observation, so as to eliminate the irregular and accidental changes. The results usually so combined are those of each month. Their number is, in general, sufficient for the purpose above-mentioned; while, on the other hand, the course of the diurnal change is sufficiently different from one month to the next, to demand a separate determination.

But in the case of deficient observations, the monthly means of the results corresponding to each hour will not exhibit, in general, the true course of the diurnal change without a correction. If a result be wanting at one hour of a day, in which all the results are much *above the mean*, it is obvious that the monthly mean corresponding to that hour will be *too small*, as compared with the means of the other hours; while, on the other hand, it will be *too great*, when all the results of the day in question are *below the mean*. The error will be greater, the greater the variation of the element observed from day to day. In the case of the atmospheric pressure it is so considerable, that the uncorrected monthly means afford no approximation to the law of the diurnal change, in the case of deficient observations.

The remedy which first suggests itself, in such a case, is to omit all the results of a day in which one or more are wanting. This process is inartificial and unsatisfactory. The *weight* of the mean is diminished in the proportion of the number of observations combined; and it is therefore important to employ all the observed results in its deduction, provided we can obtain a correction. Such a correction is easily found.

14. Let x denote the observed value of any element, at any hour on any day; and let a denote its *mean* value for that day; then

$$x = a + \xi,$$

in which ξ is the magnitude of the diurnal variation corresponding to the hour in question. Let there be n days of observation to be combined; then, summing the n results, dividing by n, and denoting the mean values by $\underline{x}$, $\underline{a}$, and $\underline{\xi}$,

$$\underline{x} = \underline{a} + \underline{\xi}.$$

Now, at any particular hour of any day, let *one* of the results be wanting; and let a' denote the mean for that day; summing the $n-1$ results,

$$S_{n-1}\,x = S_n a - a' + S_{n-1}\xi.$$

And dividing by $n-1$,

$$\underline{x} = \frac{S_n a - a'}{n-1} + \underline{\xi} = \underline{a} + \frac{\underline{a} - a'}{n-1} + \underline{\xi};$$

whence

$$\underline{x} + \frac{a' - \underline{a}}{n-1} = \underline{a} + \underline{\xi}.$$

The correction, therefore, is $+\dfrac{a' - \underline{a}}{n-1}$.

Similarly, if p results be wanting, we find

$$\underline{x} + \frac{Sa' - p\underline{a}}{n-p} = \underline{a} + \underline{\xi},$$

in which Sa' denotes the sum of the means of the days on which the deficiencies occur. Hence, the correction to be applied to the observed mean, $\underline{x}$, deduced from the $n-p$ values, is $+\dfrac{Sa' - p\underline{a}}{n-p}$.

15. The preceding correction depends, as might have been anticipated, on the difference of the daily means, for the days of deficient observations, and the *mean daily mean.* With the view of determining its probable amount, I have taken the differences between the mean of each day and the mean of the month, for the declination, the atmospheric pressure and temperature, as deduced from the

observations of the year 1843; and have calculated the square root of the mean of the squares of these differences, or the values of the expression $\sqrt{\frac{\{\Sigma(a-\underline{a})^2\}}{n}}$. The values of this quantity, which may be denominated the *mean daily error*, are the following:

Mean daily Error.

Magnetic declination, ...	Daily error = 0′·95.
Atmospheric pressure, ...	0·301.
Atmospheric temperature, ...	4°·25.

Now the mean value of n in each month (the Sundays being omitted) is 26. Hence the *mean correction*, in the case of a single deficient observation, is, for the magnetic declination, 0′·04; for the atmospheric pressure, 0·012; and for the temperature, 0°·17. In the case of the two meteorological elements, and especially in that of the atmospheric pressure, the correction is too considerable to be overlooked; in the case of the magnetic declination, and, probably, also in that of the other magnetic elements, it may be disregarded.